Black-Scholes Formulae

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1 No Arbitrage Pricing

In a general setting we are going to derive the no-arbitrage price of *any* European contingent claim. The result will be an expectation under the risk-neutral probability measure. The basic idea is that given two assets B_t (the "numeraire") and S_t, we will construct a self-financing portfolio $V_t = \phi_t S_t + \psi_t B_t$ such that at maturity, this portfolio has the same payoff as the contingent claim, where (ϕ_t, ψ_t) is the holding vector of (S_t, B_t).

Given the physical probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^T, \mathbb{P})$, imagine an European contingent claim with maturity T that pays off X. The payoff ought to be deterministic at time T, but not before T, so X is an $\mathcal{F}_T$-measurable random variable. We will replicate X with assets S_t and B_t. For simplicity set $Z_t = B_t^{-1} S_t$. A standard 4-step process to construct the self-financing portfolio is as follows:

1. Find a probability measure $\mathbb{Q}$ such that Z_t is a $\mathbb{Q}$-martingale.
2. Construct $E_t = E_{\mathbb{Q}}[B_T^{-1} X | \mathcal{F}_t]$.
3. Find an $\mathcal{F}$-adapted process ϕ_t such that $dE_t = \phi_t dZ_t$. Such a process exists by the martingale representation theorem.
4. Set $\psi_t = E_t - \phi_t Z_t$, $V_t = \phi_t S_t + \psi_t B_t$.

Now we need to verify (1) V_t is self-financing and (2) $V_T = X$, both requiring the simplified form of V_t:

$$\begin{aligned} V_t &= \phi_t S_t + \psi_t B_t \\ &= \phi_t S_t + (E_t - \phi_t Z_t) B_t \\ &= \phi_t S_t + B_t E_t - \phi_t \underbrace{B_t Z_t}_{=S_t} \\ &= B_t E_t. \end{aligned}$$

To verify (2) is easy. Indeed,

$$\begin{aligned} V_T &= B_T E_T \\ &= B_T E_{\mathbb{Q}}[B_T^{-1} X | \mathcal{F}_T] \\ &= B_T B_T^{-1} X = X. \end{aligned}$$

To verify (1), we first introduce the definition of self-financing: V_t is self-financing if and only if

$$dV_t = \phi_t dS_t + \psi_t dB_t.$$

Intuitively, the equality says that all the change of the value of V_t is from the change of S_t and B_t, but not ϕ_t and ψ_t. In other words, although we can change our portfolio anytime, it has to be selling one asset to buy another. No change of ϕ_t or ψ_t can effect the value of V_t. Initially we set up the portfolio: ϕ_0 shares of S and ψ_0 shares of B. Once it is done, no money can be put in or pulled out. Now we are ready to verify that V_t is indeed self-financing. Note that

$$\begin{aligned} dV_t &= B_t dE_t + E_t dB_t \\ &= B_t \phi_t dZ_t + (\psi_t + \phi_t Z_t) dB_t \\ &= \psi_t dB_t + \phi_t \underbrace{(B_t dZ_t + Z_t dB_t)}_{d(B_t Z_t)} \\ &= \psi_t dB_t + \phi_t dS_t. \end{aligned}$$

The definition of self-financing is satisfied.

Finally, since V_t is a self-financing portfolio that pays off X at T, by no-arbitrage principle, at all time it must have the same value as the claim that pays off X. Thus we conclude that the no-arbitrage price of this derivative is

$$\begin{aligned} V_t &= B_t E_t \\ &= B_t E_{\mathbb{Q}}[B_T^{-1} X | \mathcal{F}_t]. \end{aligned} \tag{1}$$

In particular, if B is chosen to be the cash bond, meaning $B_t = e^{rt}$ with r the risk-free interest rate, then the no-arbitrage price reduces to $e^{-r(T-t)} E_{\mathbb{Q}}[X|\mathcal{F}_t]$, the discounted expected payoff under the risk-neutral probability $\mathbb{Q}$.

The above argument is quite general since we do not assume what B_t and S_t are. The celebrated Black-Scholes formula for vanilla options can be derived from a special case of Eq. (1) where B is assumed to be the cash bond, and S_t is a log-normal diffusion process that models the dynamic of the underlying asset. Another application is to derive Margrabe's formula where both B_t and S_t are assume to follow log-normal diffusion processes.

2 The Black-Scholes Formulae

Consider the value of a vanilla put option with strike price K and maturity T at time t. Let r be the risk-free interest rate. Assume that the underlying asset follows a log-normal diffusion process

$$\begin{aligned} S_T &= S_0 e^{(r-\sigma^2/2)T + \sigma W(T)} \\ &= S_0 e^{(r-\sigma^2/2)(T-t) + (r-\sigma^2/2)t + \sigma(W(T)-W(t)) + \sigma W(t)} \\ &= S_t e^{(r-\sigma^2/2)(T-t) + \sigma(W(T)-W(t))}. \end{aligned}$$

The fair value is

$$\begin{aligned} P &= e^{-r(T-t)} E\left[(K - S_T)^+ \middle| \mathcal{F}_t\right] \\ &= e^{-r(T-t)} E\left[\left(K - S_t e^{(r-\sigma^2/2)(T-t)+\sigma(W(T)-W(t))}\right) 1_{\{K > S_T\}} \middle| \mathcal{F}_t\right], \end{aligned}$$

where

$$\begin{aligned} \{K > S_T\} &= \left\{K > S_t e^{(r-\sigma^2/2)(T-t)+\sigma(W(T)-W(t))}\right\} \\ &= \left\{\frac{\log\frac{K}{S_t} - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} > \frac{W(T)-W(t)}{\sqrt{T-t}}\right\}. \end{aligned}$$

Similarly, the fair value of a vanilla call option under the same settings is

$$\begin{aligned} C &= e^{-r(T-t)} E\left[(S_T - K)^+ \middle| \mathcal{F}_t \right] \\ &= e^{-r(T-t)} E\left[\left(S_t e^{(r-\sigma^2/2)(T-t)+\sigma(W(T)-W(t))} - K \right) 1_{\{K<S_T\}} \middle| \mathcal{F}_t \right], \end{aligned}$$

where

$$\{K < S_T\} = \left\{ \frac{\log \frac{K}{S_t} - \left(r - \frac{\sigma^2}{2} \right)(T-t)}{\sigma\sqrt{T-t}} < \frac{W(T)-W(t)}{\sqrt{T-t}} \right\}.$$

So

$$\begin{aligned} P &= e^{-r(T-t)} E\left[\left(K - S_t e^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}X} \right) 1_{\left\{ X < \frac{\log \frac{K}{S_t} - \left(r - \frac{\sigma^2}{2} \right)(T-t)}{\sigma\sqrt{T-t}} \right\}} \middle| \mathcal{F}_t \right], \\ C &= e^{-r(T-t)} E\left[\left(S_t e^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}X} - K \right) 1_{\left\{ X > \frac{\log \frac{K}{S_t} - \left(r - \frac{\sigma^2}{2} \right)(T-t)}{\sigma\sqrt{T-t}} \right\}} \middle| \mathcal{F}_t \right], \end{aligned}$$

where

$$X = \frac{W(T)-W(t)}{\sqrt{T-t}} \sim N(0,1)$$

under the filtration $\mathcal{F}_t$. Look at the put option

$$\begin{aligned} P &= e^{-r(T-t)} \int_{-\infty}^{\frac{\log \frac{K}{S_t} - \left(r - \frac{\sigma^2}{2} \right)(T-t)}{\sigma\sqrt{T-t}}} \left(K - S_t e^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}x} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\ &= K e^{-r(T-t)} N\left(-\frac{\log \frac{S_t}{K} + \left(r - \frac{\sigma^2}{2} \right)(T-t)}{\sigma\sqrt{T-t}} \right) \\ &\quad - e^{-r(T-t)} \int_{-\infty}^{\frac{\log \frac{K}{S_t} - \left(r - \frac{\sigma^2}{2} \right)(T-t)}{\sigma\sqrt{T-t}}} S_t e^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx, \end{aligned}$$

where $N(\cdot)$ is the standard normal cumulative distribution function. The second term of the right hand side is

$$\begin{aligned} & -e^{-r(T-t)} \int_{-\infty}^{\frac{\log \frac{K}{S_t} - \left(r - \frac{\sigma^2}{2} \right)(T-t)}{\sigma\sqrt{T-t}}} S_t e^{(r-\sigma^2/2)(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x^2 - 2\sigma\sqrt{T-t}x \right)}\, dx. \\ = & -e^{-r(T-t)} \int_{-\infty}^{\frac{\log \frac{K}{S_t} - \left(r - \frac{\sigma^2}{2} \right)(T-t)}{\sigma\sqrt{T-t}}} S_t e^{(r-\sigma^2/2)(T-t)+\sigma^2(T-t)/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x^2 - 2\sigma\sqrt{T-t}x + \sigma^2(T-t) \right)}\, dx. \\ = & -S_t \int_{-\infty}^{\frac{\log \frac{K}{S_t} - \left(r - \frac{\sigma^2}{2} \right)(T-t)}{\sigma\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x - \sigma\sqrt{T-t} \right)^2}\, dx. \end{aligned}$$

Use a substitution $y = x - \sigma\sqrt{T-t}$ and rewrite this integral as

$$-S_t \int_{-\infty}^{\frac{\log\frac{K}{S_t} - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} - \sigma\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy = -S_t \int_{-\infty}^{\frac{\log\frac{K}{S_t} - \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy$$

$$= -S_t N\left(-\frac{\log\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

Put together to get the Black-Scholes formula for put options

$$P = -S_t N\left(-\frac{\log\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) + Ke^{-r(T-t)} N\left(-\frac{\log\frac{S_t}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

Similarly, the price of a call option is

$$\begin{aligned} C &= e^{-r(T-t)} E\left[\left(S_t e^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}X} - K\right) 1_{\left\{X > \frac{\log\frac{K}{S_t} - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right\}} \middle| \mathcal{F}_t\right] \\ &= e^{-r(T-t)} \int_{\frac{\log\frac{K}{S_t} - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}^{\infty} \left(S_t e^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}x} - K\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\ &= e^{-r(T-t)} \int_{\frac{\log\frac{K}{S_t} - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}^{\infty} S_t e^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\ &\quad -Ke^{-r(T-t)} N\left(\frac{\log\frac{S_t}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

The first term of the right hand side can be simplified as

$$\begin{aligned} & e^{-r(T-t)} \int_{\frac{\log\frac{K}{S_t} - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}^{\infty} S_t e^{(r-\sigma^2/2)(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x^2 - 2\sigma\sqrt{T-t}x\right)}\, dx. \\ = \; & e^{-r(T-t)} \int_{\frac{\log\frac{K}{S_t} - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}^{\infty} S_t e^{(r-\sigma^2/2)(T-t)+\sigma^2(T-t)/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x^2 - 2\sigma\sqrt{T-t}x + \sigma^2(T-t)\right)}\, dx. \\ = \; & S_t \int_{\frac{\log\frac{K}{S_t} - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x - \sigma\sqrt{T-t}\right)^2}\, dx. \end{aligned}$$

Use a substitution $y = x - \sigma\sqrt{T-t}$ and rewrite this integral as

$$S_t \int_{\frac{\log\frac{K}{S_t} - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} - \sigma\sqrt{T-t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy = S_t \int_{\frac{\log\frac{K}{S_t} - \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy$$

$$= S_t N\left(\frac{\log\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

Thus we obtain the Black-Scholes formula for call options

$$C = S_t N\left(\frac{\log\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)} N\left(\frac{\log\frac{S_t}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

3 The Approximation $S_0\sigma\sqrt{T/2\pi}$

If the call is at the money and r, $T-t$ are small, then $S_t \approx Ke^{-r(T-t)}$, $\log(S_t/K) = 0$,

$$\begin{aligned} C &= S_t N\left(\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)\sqrt{T-t}\right) - Ke^{-r(T-t)}N\left(\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T-t}\right) \\ &\approx S_t\left[N\left(\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)\sqrt{T-t}\right) - N\left(\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T-t}\right)\right]. \end{aligned}$$

If $(r/\sigma+\sigma/2)\sqrt{T-t}$ is small, we can approximate $N(x)$ by its Taylor expansion at 0

$$N(x) \approx \frac{1}{2} + \frac{x}{\sqrt{2\pi}}.$$

Thus

$$\begin{aligned} C &\approx S_t\left[\left(\frac{1}{2}+\frac{\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)\sqrt{T-t}}{\sqrt{2\pi}}\right) - \left(\frac{1}{2}+\frac{\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T-t}}{\sqrt{2\pi}}\right)\right] \\ &= S_t\sigma\sqrt{\frac{T-t}{2\pi}}. \end{aligned}$$

Given the assumption $S_t \approx Ke^{-r(T-t)}$ and the put-call parity $C-P = S_t - Ke^{-r(T-t)} \approx 0$, we conclude that the approximation is for both call and put options.