BJERKSUND AND STENSLAND's CLOSED FORM VALUATION OF AMERICAN OPTIONS

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1 Bjerksund and Stensland (BS) 2002 American Option Approximation

The BS 2002 approximation for the American call option price, C_{Am}, uses a two-step optimal exercise strategy defined by two constant boundaries, I_1 and I_2, split at time $t_1 \approx 0.618T$. The price is the sum of the European price C_{BS} and the two early exercise premiums ΔC_2 and ΔC_1:

$$C_{Am}(S) = C_{BS}(S, K, T) + \Delta C_2(S) + \Delta C_1(S)$$

1.1 Derived Parameters

The formula relies on the exponent β, the positive root of a quadratic derived from the early exercise problem:

$$\beta = \frac{1}{2} - \frac{b}{\sigma^2} + \sqrt{\left(\frac{b}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

Where $b = r - q$ is the cost of carry.

The coefficients α_i are defined using the perpetual option boundary $I_\infty = \frac{\beta}{\beta-1}K$:

$$\alpha_1 = (I_1 - K)I_1^{-\beta}, \quad \alpha_2 = (I_2 - K)I_2^{-\beta}$$

2 Derivation of the Perpetual American Call Option Boundary (I_∞)

The optimal exercise boundary for the perpetual American call option is given by $I_\infty = K\left(\frac{\beta}{\beta-1}\right)$. This is derived by applying the smooth-pasting condition to the governing ordinary differential equation (ODE).

2.1 Governing ODE and General Solution

In the continuation region ($S < I_\infty$), the option value $C(S)$ satisfies the simplified Black-Scholes PDE:

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2C}{dS^2} + (r-q)S\frac{dC}{dS} - rC = 0$$

The general solution is $C(S) = AS^{\beta_1} + BS^{\beta_2}$, where β_1 and β_2 are the roots of the indicial equation. We define β as the positive root (β_1):

$$\beta = \frac{1}{2} - \frac{r-q}{\sigma^2} + \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

Since $C(S) \to 0$ as $S \to 0$, the term associated with the negative root (β_2) is dropped. The solution in the continuation region is:

$$C(S) = AS^\beta \quad \ldots\text{(Eq. 1)}$$

2.2 Boundary Conditions at $S = I_\infty$

The optimal boundary I_∞ must satisfy two conditions:

2.2.1 Value Matching Condition

The option value must equal its intrinsic value at the boundary:

$$C(I_\infty) = I_\infty - K$$

Substituting (Eq. 1):

$$AI_\infty^\beta = I_\infty - K \quad \ldots\text{(Eq. 2)}$$

2.2.2 Smooth Pasting Condition

The derivative of the option price must match the derivative of the intrinsic value (which is 1):

$$\left.\frac{dC}{dS}\right|_{S=I_\infty} = 1$$

Substituting the derivative $\frac{dC}{dS} = A\beta S^{\beta-1}$:

$$A\beta I_\infty^{\beta-1} = 1 \quad \ldots\text{(Eq. 3)}$$

2.3 Solving for I_∞

1. Solve (Eq. 3) for A:

$$A = \frac{1}{\beta I_\infty^{\beta-1}}$$

2. Substitute A into (Eq. 2):

$$\left(\frac{1}{\beta I_\infty^{\beta-1}}\right) I_\infty^\beta = I_\infty - K$$

3. Simplify the left-hand side (LHS):

$$\frac{I_\infty}{\beta} = I_\infty - K$$

4. Solve for I_∞:

$$K = I_\infty - \frac{I_\infty}{\beta} = I_\infty \left(1 - \frac{1}{\beta}\right) = I_\infty \left(\frac{\beta - 1}{\beta}\right)$$

Rearranging the final expression yields the formula for the optimal perpetual boundary:

$$I_\infty = K \left(\frac{\beta}{\beta - 1}\right)$$

2.4 Early Exercise Boundaries (I_1 and I_2)

The boundaries are approximated using empirical functions h_1 and h_2 applied to the perpetual boundary I_∞:

$$\begin{cases} I_2 = I_\infty + (K - I_\infty)e^{h_2(T)} & \text{for time } t \in (t_1, T] \\ I_1 = I_\infty + (K - I_\infty)e^{h_1(t_1)} & \text{for time } t \in [0, t_1] \end{cases}$$

2.5 Boundary Functions h_1 and h_2

The functions h_1 and h_2 are empirically derived decay factors used to calculate the two constant early exercise boundaries, I_1 and I_2, respectively.

2.5.1 Function h_1 (for boundary I_1)

The functions h_1 and h_2 incorporate the cost of carry (b), volatility (σ), and the relevant time to maturity (t_1 or T)

$$h_1 = -\left(bt_1 + 2\sigma\sqrt{t_1}\right)\left(\frac{K^2}{(I_\infty - K)I_\infty}\right)$$

2.5.2 Function h_2 (for boundary I_2)

$$h_2 = -\left(bT + 2\sigma\sqrt{T}\right)\left(\frac{K^2}{(I_\infty - K)I_\infty}\right)$$

3 Derivation of Boundary Functions h_1 and h_2 (BS 2002)

The functions h_1 and h_2 were obtained via an **empirical and heuristic approach**, not a direct analytical solution of the Black-Scholes PDE. They serve as **decay factors** to transition the optimal exercise boundary from the perpetual case (I_∞) to the strike price (K) as time to maturity (T) decreases.

3.1 Modeling the Boundary Decay

The authors modeled the time-dependent boundary $I(T)$ as an interpolation between the perpetual boundary I_∞ and the strike K:

$$I(T) = I_\infty + (K - I_\infty)e^h$$

This requires the factor e^h to approach 1 as $T \to \infty$ and approach 0 as $T \to 0$.

3.2 Heuristic Functional Form

The general functional form for $h(T)$ was empirically chosen and calibrated to best fit accurate numerical solutions across various parameters. It incorporates both the drift (bT) and diffusion ($\sigma\sqrt{T}$) effects:

$$h(T) = -\left(bT + 2\sigma\sqrt{T}\right) \times \left(\frac{K^2}{(I_\infty - K)I_\infty}\right)$$

3.3 Assignment to h_1 and h_2

The two boundaries are defined by applying the function h to the relevant time parameters:

3.3.1 Function h_2 (Uses total time T)

This governs the boundary I_2 in the second period $(t_1, T]$.

$$h_2 = -\left(bT + 2\sigma\sqrt{T}\right)\left(\frac{K^2}{(I_\infty - K)I_\infty}\right)$$

3.3.2 Function h_1 (Uses intermediate time t_1)

This governs the boundary I_1 in the first period $[0, t_1]$. Since $t_1 < T$, h_1 is less negative than h_2, correctly producingg a higher boundary ($I_1 > I_2$):

$$h_1 = -\left(bt_1 + 2\sigma\sqrt{t_1}\right)\left(\frac{K^2}{(I_\infty - K)I_\infty}\right)$$

Where $I_\infty = K\left(\frac{\beta}{\beta-1}\right)$ and $b = r - q$.

3.4 Parameter Definitions

- T: Total time to maturity.
- t_1: Intermediate time split (typically $t_1 \approx 0.618T$).
- K: Strike price.
- b: Cost of carry ($b = r - q$).
- σ: Volatility.
- β: Exponent derived from the early exercise problem.

3.5 Early Exercise Premium Components

The premium terms are defined using specialized functions $\Phi(\cdot)$, $\Psi(\cdot)$, and $\Omega(\cdot)$ that rely on the univariate $N(\cdot)$ and bivariate $M(\cdot, \cdot, \rho)$ cumulative normal distribution functions.

3.5.1 Premium for Period 2 (ΔC_2)

$$\Delta C_2(S) = \alpha_2 S^\beta \left[1 - \Psi(S, t_1, \beta, I_2, I_2)\right] + \Phi(S, t_1, I_2, I_2) - K\Phi(S, t_1, 0, I_2, I_2)$$

3.5.2 Premium for Period 1 (ΔC_1)

$$\Delta C_1(S) = \alpha_1 \Psi(S, t_1, \beta, I_1, I_2) - \alpha_2 \Omega(S, T, \beta, I_1, I_2, I_1, t_1) + \sum_{j \in \{1,0\}} (-1)^{1-j} K^j \Omega(S, T, 1-j, I_1, I_2, I_1, t_1)$$

3.6 American Put Option

The American put price P_{Am} is obtained by applying the standard put-call transformation:

$$P_{Am}(S, K, T, r, b, \sigma) = C_{Am}(K, S, T, r - b, -b, \sigma).$$

4 Key Idea and Derivation for the premium components

The Bjerksund and Stensland (2002) model provides a closed-form approximation for the American option value, $C_{Am}(S)$, by replacing the complex, time-varying optimal exercise boundary with a two-step, constant-boundary strategy.

5 The Exercise Strategy and Decomposition

The method divides the total time to maturity T into two periods, separated by an intermediate time split, t_1. The specific boundaries used for exercise are:

- I_1: The flat exercise boundary for the first period, $[0, t_1]$.
- I_2: The flat exercise boundary for the second period, $[t_1, T]$.

This strategy assumes exercise occurs if the stock price S_τ hits I_1 in the first period, or I_2 in the second period (having survived I_1).

The option value $C(S)$ is decomposed into four mutually exclusive, analytically solvable components based on the time the option is exercised, τ.

$$C(S) = \underbrace{\mathbb{E}_0[e^{-r\tau}(I_1 - K)\mathbb{I}(0 < \tau < t_1)]}_{\text{1. Rebate at } I_1 \text{ in } [0,t_1]} \tag{1}$$

$$+ \underbrace{\mathbb{E}_0[e^{-rt_1}(S_{t_1} - K)\mathbb{I}(\tau = t_1)]}_{\text{2. Exercise at } t_1 \text{ (if } I_2 \le S_{t_1} < I_1)} \tag{2}$$

$$+ \underbrace{\mathbb{E}_0[e^{-r\tau}(I_2 - K)\mathbb{I}(t_1 < \tau < T)]}_{\text{3. Rebate at } I_2 \text{ in } [t_1,T]} \tag{3}$$

$$+ \underbrace{\mathbb{E}_0[e^{-rT}(S_T - K)^+\mathbb{I}(\tau = T)]}_{\text{4. European Payoff at } T} \tag{4}$$

Note: $\mathbb{I}(\cdot)$ *is the indicator function.*

6 Key Derivation Functions

The closed-form nature of the solution is achieved by expressing the expectations above using specific functions:

- $\varphi(\cdot)$: Used for expectations over a single time period, based on the Reflection Principle (single barrier). It is expressed using the standard univariate normal cumulative distribution function $N(\cdot)$.
- $\Psi(\cdot)$: Used for expectations over the full time span $[0, T]$ that involve the path staying below I_1 in the first period and I_2 in the second period. This is solved using the Method of Images and is expressed using the Bivariate Normal Distribution function $M(\cdot, \cdot; \rho)$.

The coefficient β is a key derived parameter, the positive root of a quadratic derived from the perpetual option problem:

$$\beta = \frac{1}{2} - \frac{b}{\sigma^2} + \sqrt{\left(\frac{b}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{5}$$

The coefficients α_1 and α_2 are defined using the perpetual option boundary concept, $I_\infty = \frac{\beta}{\beta-1}K$:

$$\alpha_1 = (I_1 - K)I_1^{-\beta} \quad \text{and} \quad \alpha_2 = (I_2 - K)I_2^{-\beta} \tag{6}$$

7 The Final Closed-Form Formula

The sum of the four expected values (rearranged and simplified into the form of Proposition 1 from the paper, but using the I_1, I_2, t_1 notation) is:

$$\begin{aligned}
C(S) = {} & \alpha_1 S^\beta - \alpha_1 \varphi(S, t_1|\beta, I_1, I_1) \\
& + \varphi(S, t_1|1, I_1, I_1) - \varphi(S, t_1|1, I_2, I_1) - K\varphi(S, t_1|0, I_1, I_1) + K\varphi(S, t_1|0, I_2, I_1) \\
& + \alpha_2 \varphi(S, t_1|\beta, I_2, I_1) - \alpha_2 \Psi(S, T|\beta, I_2, I_1, I_2, t_1) \\
& + \Psi(S, T|1, I_2, I_1, I_2, t_1) - \Psi(S, T|1, K, I_1, I_2, t_1) \\
& - K\Psi(S, T|0, I_2, I_1, I_2, t_1) + K\Psi(S, T|0, K, I_1, I_2, t_1)
\end{aligned}$$