# Forward Measure and Bond Option Pricing in the Hull–White Model

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## 1 Hull–White Short-Rate Model

We begin with the one-factor Hull–White model under the risk-neutral measure $\mathbb{Q}$:

$$dr_t = (\theta(t) - ar_t)\, dt + \sigma\, dW_t^{\mathbb{Q}}, \tag{1}$$

where $a$ is the mean-reversion speed, $\sigma$ is the short-rate volatility, and $\theta(t)$ ensures the model fits the initial yield curve.

Zero-coupon bond prices under $\mathbb{Q}$ admit the affine form

$$P(t,T) = A(t,T) \exp\big( - B(t,T) r_t \big), \tag{2}$$

with deterministic functions

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}, \tag{3}$$

$$A(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left( \frac{\sigma^2}{4a^3} (1 - e^{-a(T-t)})^2 (1 - e^{-2at}) \right). \tag{4}$$

Since $r_t$ is Gaussian, the exponent $-B(t,T)r_t$ is Gaussian, and $P(t,T)$ is *exponential Gaussian.*

## 2 Motivation for the Forward Measure

Consider a European call option maturing at time $t$ on a zero-coupon bond maturing at $T > t$, with payoff

$$\max\big(P(t,T) - K,\, 0\big). \tag{5}$$

Instead of discounting the payoff using the money-market account, it is natural to use the bond $P(t,T)$ itself as numéraire. This leads to the *$T$-forward measure* $\mathbb{Q}^T$, under which any price divided by $P(t,T)$ is a martingale.

## 3 Definition of the $T$-Forward Measure

Let $B_t = \exp\left(\int_0^t r_s\, ds\right)$ be the money-market numéraire. We change numéraire from $B_t$ to the zero-coupon bond $P(t,T)$.

The Radon–Nikodym derivative defining the $T$-forward measure is

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{B_0\, P(t,T)}{B_t\, P(0,T)}. \tag{6}$$

Under $\mathbb{Q}^T$,

$$\frac{P(t,T')}{P(t,T)} \quad \text{is a martingale for all } T' > t. \tag{7}$$

This property is fundamental for deriving the closed-form bond option price.

## 4 Bond Dynamics Under the Forward Measure

From the affine bond-price formula,

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}, \tag{8}$$

and the Hull–White short-rate dynamics, Itô's lemma implies that under $\mathbb{Q}$,

$$\frac{dP(t,T)}{P(t,T)} = \mu_P(t)\, dt + \sigma_P(t)\, dW_t^{\mathbb{Q}}, \qquad \sigma_P(t) = -\sigma B(t,T). \tag{9}$$

When changing to $\mathbb{Q}^T$, Girsanov's theorem alters the drift but preserves the volatility. Since $P(t,T)$ becomes the numéraire, its discounted value is constant, and the bond-price process has zero drift under $\mathbb{Q}^T$:

$$\frac{dP(t,T)}{P(t,T)} = -\sigma B(t,T)\, dW_t^{\mathbb{Q}^T}. \tag{10}$$

Thus the log-price satisfies

$$d\ln P(t,T) = -\sigma B(t,T)\, dW_t^{\mathbb{Q}^T} - \frac{1}{2}\sigma^2 B(t,T)^2 dt. \tag{11}$$

Integrating from 0 to $t$ gives

$$\ln P(t,T) = \ln P(0,T) - \int_0^t \sigma B(s,T)\, dW_s^{\mathbb{Q}^T} - \frac{1}{2}\int_0^t \sigma^2 B(s,T)^2 ds, \tag{12}$$

which is normally distributed.

## 5 Forward-Measure Variance

The variance of $\ln P(t,T)$ under $\mathbb{Q}^T$ is

$$\sigma_P^2 = \int_0^t \sigma^2 B(s,T)^2 \, ds, \tag{13}$$

which evaluates to

$$\sigma_P^2 = \frac{\sigma^2}{2a^3}(1 - e^{-a(T-t)})^2(1 - e^{-2at}). \tag{14}$$

Thus $P(t,T)$ is *lognormally distributed* under the $T$-forward measure.

## 6 Closed-Form Bond Option Price

Since the payoff is the expectation of a lognormal variable under $\mathbb{Q}^T$, we obtain a Black-style formula. Define

$$d_1 = \frac{\ln\left(\frac{P(0,T)}{KP(0,t)}\right) + \frac{1}{2}\sigma_P^2}{\sigma_P}, \qquad d_2 = d_1 - \sigma_P. \tag{15}$$

Then the European call price is

$$C = P(0,T)\, N(d_1) - K\, P(0,t)\, N(d_2), \tag{16}$$

with $N(\cdot)$ the standard normal CDF.

## 7 Summary

The steps for using the forward measure in the Hull–White model are:

1. Start with the short-rate SDE under $\mathbb{Q}$.
2. Use the zero-coupon bond $P(t,T)$ as the numéraire.
3. Change measure using the Radon–Nikodym derivative.
4. Obtain deterministic bond volatility under $\mathbb{Q}^T$.
5. Deduce that the bond price is lognormal at option maturity.
6. Use the Black formula to compute the closed-form bond option price.

The affine structure of the Hull–White model makes each of these steps explicit and analytically tractable.

## 8 Hull–White Model Under the Risk-Neutral Measure

The one-factor Hull–White short-rate model under the risk-neutral measure $\mathbb{Q}$ is

$$dr_t = (\theta(t) - ar_t)\,dt + \sigma\,dW_t^{\mathbb{Q}}, \tag{17}$$

where $a$ is the mean-reversion speed, $\sigma$ is the volatility, and $\theta(t)$ is chosen such that the model fits the initial term structure.

The money-market numéraire is

$$B_t = \exp\left(\int_0^t r_s\,ds\right). \tag{18}$$

Zero-coupon bond prices are affine in the short rate:

$$P(t,T) = A(t,T)\exp(-B(t,T)r_t)\,, \tag{19}$$

with

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}, \tag{20}$$

$$A(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left(\frac{\sigma^2}{4a^3}(1 - e^{-a(T-t)})^2(1 - e^{-2at})\right). \tag{21}$$

## 9 Motivation for the $T$-Forward Measure

For pricing a European option with maturity $t$ on a zero-coupon bond with maturity $T > t$, it is natural to rewrite the payoff

$$\max(P(t,T) - K,\, 0)$$

in terms of a measure under which the bond $P(t,T)$ serves as the numéraire.

The $T$-*forward measure* $\mathbb{Q}^T$ is defined so that

$$\frac{P(t,T')}{P(t,T)} \quad \text{is a martingale under } \mathbb{Q}^T \quad \text{for all } T' > t. \tag{22}$$

## 10 Definition of the Measure Change

We change numéraire from $B_t$ to $P(t,T)$. The Radon–Nikodym derivative is

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{\frac{P(t,T)}{B_t}}{\frac{P(0,T)}{B_0}} = \frac{B_0\,P(t,T)}{B_t\,P(0,T)}. \tag{23}$$

This ensures that $P(t,T)$ becomes the numéraire under $\mathbb{Q}^T$, exactly as $B_t$ was the numéraire under $\mathbb{Q}$.

## 11 Brownian Motion Under the $T$-Forward Measure

Let $W_t^{\mathbb{Q}}$ be the Brownian motion under $\mathbb{Q}$ in the short-rate SDE (17). By Girsanov's theorem, the Brownian motion changes according to

$$dW_t^{\mathbb{Q}^T} = dW_t^{\mathbb{Q}} + \lambda_t \, dt, \tag{24}$$

where $\lambda_t$ is the *market price of risk associated with the new numéraire.*

To compute $\lambda_t$, differentiate the bond price:

$$P(t,T) = A(t,T) \exp(-B(t,T) r_t).$$

Applying Itô's lemma yields the volatility of the bond under $\mathbb{Q}$:

$$\frac{dP(t,T)}{P(t,T)} = \mu_P(t) \, dt - \sigma B(t,T) \, dW_t^{\mathbb{Q}}. \tag{25}$$

Because the drift must be zero under the measure where $P(t,T)$ is the numéraire, Girsanov's theorem implies

$$\lambda_t = -\sigma B(t,T). \tag{26}$$

Thus under the $T$-forward measure,

$$dW_t^{\mathbb{Q}^T} = dW_t^{\mathbb{Q}} - \sigma B(t,T) \, dt. \tag{27}$$

## 12 Bond Dynamics Under the $T$-Forward Measure

Under $\mathbb{Q}^T$, the bond price volatility is unchanged, but the drift disappears:

$$\frac{dP(t,T)}{P(t,T)} = -\sigma B(t,T) \, dW_t^{\mathbb{Q}^T}. \tag{28}$$

Hence the log-price satisfies

$$d \ln P(t,T) = -\sigma B(t,T) \, dW_t^{\mathbb{Q}^T} - \frac{1}{2} \sigma^2 B(t,T)^2 dt. \tag{29}$$

Integrating gives

$$\ln P(t,T) = \ln P(0,T) - \int_0^t \sigma B(s,T) \, dW_s^{\mathbb{Q}^T} - \frac{1}{2} \int_0^t \sigma^2 B(s,T)^2 \, ds. \tag{30}$$

Because the integral of $B(s,T)$ is deterministic, $\ln P(t,T)$ is normally distributed.

## 13 Forward-Measure Variance

The variance of $\ln P(t,T)$ under $\mathbb{Q}^T$ is explicitly

$$\sigma_P^2 = \int_0^t \sigma^2 B(s,T)^2 \, ds, \tag{31}$$

which evaluates to

$$\sigma_P^2 = \frac{\sigma^2}{2a^3} (1 - e^{-a(T-t)})^2 (1 - e^{-2at}). \tag{32}$$

Thus $P(t,T)$ is lognormal at time $t$ under $\mathbb{Q}^T$, enabling closed-form bond option pricing.

# 14 Jamshidian's Trick

Consider a European option on a coupon bond with maturity $S$ and coupons $\{c_i\}$ at times $\{T_i\}$:

$$H = \left( \sum_i c_i P(T, T_i) - K \right)^+ .$$

Jamshidian's theorem states:

- Under a one-factor short-rate model (such as Hull–White),
- There exists a unique short rate level $r$ solving

$$\sum_i c_i P^{\mathrm{HW}}(T, T_i; r^{)=K.}$$

- The option price decomposes into a sum of zero-coupon bond options:

$$V_t = \sum_i c_i \, C^{\mathrm{ZCB}}(t; T, T_i, K_i),$$

  where each strike $K_i$ is

$$K_i = P^{\mathrm{HW}}(T, T_i; r^{).}$$

## 14.1 Why the Trick Works

Since the one-factor short rate determines all bond prices through

$$P(T, T_i) = A(T, T_i) e^{-B(T, T_i) r_T},$$

the coupon bond payoff is a monotone function of $r_T$:

$$\sum_i c_i P(T, T_i) \quad \text{is strictly decreasing in } r_T.$$

Thus there is a unique $r$ such that the coupon bond price equals $K$. The event

$$\sum_i c_i P(T, T_i) > K$$

is equivalent to

$$r_T < r^{\cdot}$$

Hence the payoff decomposes into

$$H = \sum_i c_i \left( P(T, T_i) - K_i \right)^+ .$$

Since each $P(T, T_i)$ is lognormal under the $T$-forward measure, each term is a standard zero-coupon bond option.

## 15 Final Pricing Formula

The price at time $t$ is

$$V_t = P(t,T) \sum_i c_i \, \mathbb{E}_{\mathbb{Q}^T} \left[ (P(T,T_i) - K_i)^+ \mid \mathcal{F}_t \right].$$

Each expectation is given by the Hull–White bond option closed form:

$$C^{\text{ZCB}}(t;T,T_i,K_i) = P(t,T_i)N(d_1) - K_i \, P(t,T)N(d_2),$$

where

$$d_{1,2} = \frac{\ln\left(\frac{P(t,T_i)}{K_i P(t,T)}\right) \pm \frac{1}{2}\sigma_{t,T,T_i}^2}{\sigma_{t,T,T_i}},$$

and $\sigma_{t,T,T_i}$ is the forward-measure bond volatility.

## 16 Conclusion

The $T$-forward measure arises naturally in the Hull–White model by choosing the zero-coupon bond $P(t,T)$ as the numéraire. The Radon–Nikodym derivative (23) changes the measure from $\mathbb{Q}$ to $\mathbb{Q}^T$, the Brownian motion shifts accordingly, and the resulting bond dynamics have zero drift and deterministic volatility. This yields a lognormal bond price under $\mathbb{Q}^T$, which is the basis for the closed-form European bond option formula.